STABILITY OF THE FLOW OF A VISCOUS FLUID AROUND AN ELASTIC BODY

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It is well known that in the flow of a viscous incompressible fluid past a stationary wall there exists a traditional curve of neutral stability. This curve bounds a zone of instability for a Tollmien–Schlichting wave in the plane of values of the Reynolds number and wave number. There are a number of papers (see, e.g., [1-3]) wherein a study was made of the effect of pliability of the wall on the position of this neutral stability curve; studies have also been made (see [4, 5]) of the stability of an elastic body (of a halfspace) in the flow of an ideal fluid (compressible and incompressible). In particular, notice has also been taken of an increase in the stability of a laminar flow (a delay in the transition to a turbulent regime) and the appearance of new forms of instability of Kelvin–Helmholtz type or of hydroelastic flutter.

In the present paper we examine the combined oscillations of an elastic layer (of a halfspace) and the viscous incompressible fluid flow over it. We find, in this connection, that it is possible to have a new type of instability, a limiting state of which, for a decrease in the relative density of the flow, is a Rayleigh wave in the elastic body.

In domain $G_2$ (upper halfspace) we consider a two-dimensional (in the x0z plane) flow of an incompressible viscous fluid flowing over an elastic lower halfspace (domain $G_1$) in which, under the action of a variable pressure, surface waves arise. Flow of the fluid is described by the linearized Navier–Stokes equations with velocity components $v_x$ and $v_z$ and pressure p, which represent small deviations from the main unperturbed flow with parameters $U = U(z)$, $V = 0$, $P = P(x)$.

Let $U_0$ be the speed of the unperturbed flow along the x axis at infinity; let $\rho_2$ be the density of the fluid. Velocity components are related to the stream function $\psi$ through the relations $v_x = \partial\psi/\partial z$, $v_z = -\partial\psi/\partial x$.

Assuming only flow perturbations periodic in the x direction, we represent the stream function in the running wave form

$$\psi(x,\ z,\ t) = \varphi(z)e^{i\alpha x - i\omega t},$$

where $\alpha$ is a dimensionless wave number; $\omega$ is the frequency of the oscillations. All the remaining variable parameters of the flow and of the elastic body over which the flow takes place will be assumed proportional to a harmonic of the same form

$$\theta(x,\ z,\ t) = \vartheta(z)e^{i\alpha x - i\omega t}. \tag{1}$$

In domain $G_2$ we also consider a boundary layer thickness $\delta_0$, such that $U(\delta_0) = U_0$. The problem for perturbation of velocities may be reduced to the well-known Orr–Sommerfeld equation

$$(U - c)(\varphi'' - \alpha^2\varphi) - U''\varphi = -\frac{i}{\alpha \operatorname{Re}}(\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi). \tag{2}$$

Here $c = \omega/\alpha$ is the phase velocity; $\operatorname{Re} = U_0\delta/\nu$ is the Reynolds number; $\delta = \delta_0/q$ ($q = 6.2$ according to [6]); $\nu$ is the coefficient of kinematic viscosity; $U_0$ and $\delta$ are taken to be characteristic magnitudes (of velocity and length).

Conditions for decay of the perturbations at infinity

$$\varphi \to 0,\ \varphi' \to 0\ (z \to \infty) \tag{3}$$

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are more suitably replaced (for technical reasons) by equivalent conditions on the finite boundary $z = \delta_0$; this was done, as in [7], starting from the following considerations. When the speed of the principal flow is constant ($U_0 = 1$) for $z \geq \delta_0$, then $U'' = 0$. Moreover, Eq. (2) has a solution satisfying the conditions (3):

$$\varphi = C_1 e^{-\gamma z} + C_2 e^{-\alpha z} \quad (\gamma = \sqrt{\alpha(i \operatorname{Re}(1-c) + \alpha)}\ ).$$

The equation reconstructed from this solution

$$\varphi'' + (\alpha + \gamma)\varphi' + \alpha\gamma\varphi = 0 \tag{4}$$

and the equation resulting from differentiating it

$$\varphi''' + (\alpha + \gamma)\varphi'' + \alpha\gamma\varphi' = 0 \tag{5}$$

are, in fact, the desired "soft" boundary conditions indicating a smooth transition for $z = \delta_0$ of the solution of Eq. (2) in the boundary layer into a solution outside the boundary layer, which satisfies conditions (3) at infinity.

On the boundary of the flow with the elastic body conditions for adhesion to the moving boundary w are satisfied:

$$\frac{\partial\varphi}{\partial z} = v_x^0, \quad -i\alpha\varphi = v_z^0 \quad (z = w) \tag{6}$$

($v_x^0 = u_x$, $v_z^0 = u_z$ are the displacement rates on the boundary of the elastic body). Rates on the fixed boundary can be represented in the form of expansions

$$(v_z)_{z=0} = v_z^0 - w\left\{\frac{\partial v_z}{\partial z}\right\}_{z=w} + O(w^2),$$

$$(v_x)_{z=0} = v_x^0 - w\left\{\frac{\partial v_x}{\partial z}\right\}_{z=w} + O(w^2).$$

Taking into account the type of solution assumed in Eq. (1), it follows that $w \sim v_z^0/\omega$. Assuming the frequency $\omega$ to be bounded in absolute value (from above and below) and assuming $v_z$, $v_x \ll 1$, we can neglect the products $w(\partial v_z/\partial z)_w$ and $w(\partial v_x/\partial z)_w$ in these expansions since $v_z$, $\partial v_z/\partial z$, $\partial v_x/\partial z$, in the linear setting of the problem adopted here, may be considered small. Hence the conditions (6) can be carried down to the unperturbed boundary $z = 0$.

To determine $v_x^0$ and $v_z^0$ it is necessary to consider the problem involving propagation of a wave into the elastic halfspace $G_1$ under the action of a pressure wave generated in the domain $G_2$.

In accordance with [8], for the domain $G_2$ we write the equation of motion

$$\frac{\partial^2 u_l}{\partial t^2} - c_l^2 \Delta u_l = 0, \quad \frac{\partial^2 u_t}{\partial t^2} - c_t^2 \Delta u_t = 0; \tag{7}$$

$$\operatorname{div} u_t = 0, \ \operatorname{rot} u_l = 0, \tag{8}$$

where $u_\ell$ and $u_t$ are dimensionless displacement vectors; $c_\ell$ and $c_t$ are compression and shear wave deformation rates. Physical displacements in the direction of the x and z axes

$$u_x = u_{lx} + u_{tx}, \ u_z = u_{lz} + u_{tz} \tag{9}$$

arise in the domain $G_1$ as the result of the action of perturbations in the flow. Unperturbed background flow in domain $G_2$ gives rise in domain $G_1$ to constant background deformations, which we shall not consider here.

Assuming that $1/\sqrt{\operatorname{Re}} \ll 1$, $\alpha < 1$ or $\alpha \sim 1$ on the boundary $z = 0$, the tangential stresses in the viscous fluid may be assumed to be negligibly small, i.e.,

$$\sigma_{xz} = 0. \tag{10}$$

Similarly, we can dispense with the viscous term in the normal stress

$$\sigma_{zz} = -p(x, 0, t)/\rho \tag{11}$$

($\sigma_{xz}$, $\sigma_{zz}$ are stresses in the elastic body). At an infinite distance from the flowed-over boundary (for $z = -\infty$) we take the decay conditions

$$u_x = 0, \; u_z = 0. \tag{12}$$

The solutions $u_{\ell x}$, $u_{\ell z}$, $u_{tx}$, $u_{tz}$ [in the form of the wave (1)] of the system (7), (8) with the boundary conditions (10)-(12) enable us to find $u_x$ and $u_z$ from relations (9) and, hence, $v_x^0$ and $v_z^0$ from equations (6) in the form

$$v_z^0 = -i\omega a p, \quad v_x^0 = -i\omega b p \quad \text{for} \quad z = 0 \tag{13}$$

[p(x, 0, t) is the pressure of the perturbed flow on the boundary];

$$a = \frac{c^2\rho}{c_t^4\alpha\Delta}, \quad b = -\frac{i\rho\left(1-\varkappa_t^2\right)}{c_t^2\Delta}, \quad \varkappa_t = \sqrt{1-\left(\frac{c}{c_t}\right)^2}; \tag{14}$$

$$\Delta = \left(1+\varkappa_t^2\right)^2 - 4\varkappa_t^2, \quad \rho = \rho_2/\rho_1. \tag{15}$$

Expressing p on the boundary from the linearized Navier—Stokes equations, we can obtain the following from relations (13):

$$ac\left[\frac{\varphi'''(0)}{i\alpha\,\mathrm{Re}} + \left(c + \frac{i\alpha}{\mathrm{Re}}\right)\varphi'(0) + U'(0)\,\varphi(0)\right] - \varphi(0) = 0; \tag{16}$$

$$\varphi'(0) + i\alpha\frac{b}{a}\varphi(0) = 0. \tag{17}$$

Thus we have posed the following homogeneous boundary value problem: the Orr—Sommerfeld equation (2) with boundary conditions (4), (5) for $z = \delta_0$ and relations (16), (17) for $z = 0$. Here $c = \omega/\alpha$ is a characteristic value whose imaginary part $c_i$ defines the decrement (increment for $c_i < 0$) of decrease (of increase) of the wave amplitude, while the real part $c_r$ is the phase velocity of this wave.

If the expression $\Delta = 0$, where $\Delta$ is given by relation (15), is considered as an equation in $\xi = c/c_t$, its solution $\xi = \xi_0 = 0.955$ gives the phase velocity of the Rayleigh surface wave in the elastic body with a free boundary [8].

Thus, as can be seen from relations (14) and (15), if as $\rho \to 0$, $\xi \to \xi_0$ simultaneously, pulsations in the velocity on the boundary w may remain finite and nonzero. This corresponds to a Rayleigh wave in both the elastic body and in the "degenerate" flow ($\rho \to 0$). If as $\rho \to 0$, $\xi$ tends towards a limit other than $\xi_0$, so that $\Delta \neq 0$, the pulsations in the velocity on the boundary w vanish. Such a solution is equivalent in the limit to a Tollmien—Schlichting wave in a flow over a fixed boundary.

For a numerical study of the spectrum of the homogeneous problem (2), (4), (5), (16), (17) we employ here a method based on spline-collocation, developed in [9] for similar purposes, from which (with the kind consent of the author) we excerpted the program whose modification was used to obtain the numerical results presented here.

The calculations presented below are of a purely methodical nature and are intended to illustrate the interaction of a flow with surface waves of a solid. They correspond roughly to flow over a body covered by a fairly thick layer of a rubbery material (for rubber, e.g., shear wave speed ∿20 m/sec, density ∿1000 kg/m³). The parameters used in our calculations, $c_t = 0.5$ ($c_t$ is referred to $U_0$) and $\rho = 1$, are then stipulated by flow of water with speed ∿40 m/sec. We assumed that $c_t \ll c_\ell$, so that the ratio $c_t/c_\ell$ can be neglected. The quantities $\alpha_* = 1.72\alpha$, $\mathrm{Re}_* = 1.72\mathrm{Re}$, namely, the wave number and the Reynolds number, are referred to the displacement thickness.

Numerically, we singled out and studied two types of solutions: the traditional Tollmien—Schlichting solution and the Rayleigh solution. Results of the first solution are shown in Fig. 1, where the neutral stability curves 1-4 correspond to the values $\rho = 0$; 0.01;

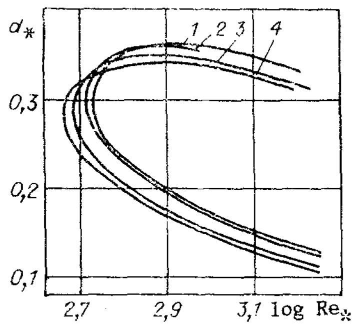


Fig. 1

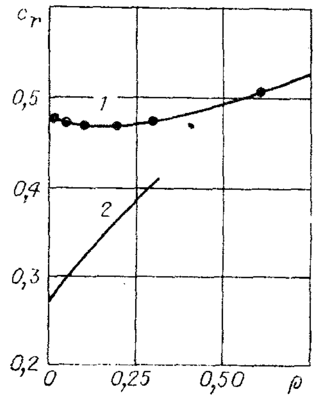


Fig. 2

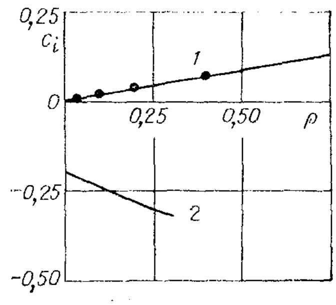


Fig. 3

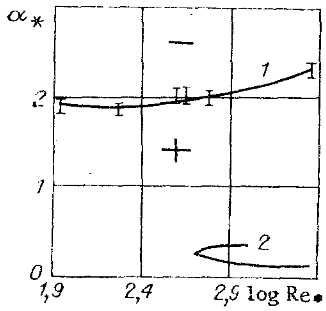


Fig. 4

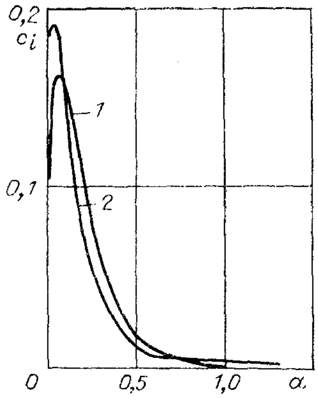


Fig. 5

0.05; 0.07. Visible here are the continuous smooth approaches of these curves, as $\rho \to 0$, to the well-known situation for the case of flow over a fixed surface.

Figure 2 shows the dependence of the phase velocity $c_r$ on the density ratio $\rho$ for Re = 360, $\alpha$ = 0.013 for Rayleigh and Tollmien–Schlichting solutions (curves 1 and 2, respectively). It is seen that as $\rho \to 0$ (this is characterized by a clustering of the computed points) $c_r \to 0.477 = c_t\xi_0$. This is precisely equal to the phase velocity of the free (i.e., without pressure) surface of the Rayleigh wave.

Figure 3 shows the dependence on $\rho$ of the increment of increase of the Rayleigh wave and the decrement of decay of the Tollmien–Schlichting wave (curves 1 and 2). It is characteristic (for curve 1) that as $\rho \to 0$, $c_i \to 0$ (observe the clustering of the computed points). Consequently, for the Rayleigh solution we obtain a wave whose phase velocity is equal to the phase velocity of free elastic oscillations, while the increment of increase is equal to zero. Hence, this confirms the fact that the first type of solution constitutes a reorganization of the free oscillations of the surface of the elastic body (Rayleigh wave) under the action of the viscous flow.

Figure 4 displays the stability in the plane of the parameters $\alpha_*$, $Re_*$ for the Rayleigh solution (Curve 1). The little vertical dashes denote limits to the position of the approximately calculated curve. The domain of stability is indicated by a minus sign; that of stability with a plus sign. For comparison we show the Tollmien–Schlichting stability loop (curve 2). A feature of the Rayleigh solution is that the domain of instability extends for smaller $Re_*$ in comparison with the domain bounded by this loop. Behavior of curve 1 for $\log Re_* < 1.9$ was not investigated on account of technical computational difficulties connected with the computational method adopted in our work.

Figure 5 shows the dependence of the increment of increase $c_i$ on the wave number $\alpha$ for Re = 170; 660 (curves 1 and 2) for the Rayleigh type of solution. It should be noted that the most unstable are the long wave perturbations (with wave length on the order of three to ten boundary layer thicknesses).

The numerical results and conclusions obtained here can turn out to be useful in formulating studies (experimental, analytical, or numerical) with the aim of generating specified waves of finite amplitude on the surface of elastic bodies by choosing material parameters in accordance with the flow parameters.

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